Fringe analysis of synchronized parallel algorithms on 2–3 trees*

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Abstract. We are interested in the *fringe* analysis of synchronized parallel insertion algorithms on 2–3 trees, namely the algorithm of W. Paul, U. Vishkin and H. Wagener (PVW). This algorithm inserts k keys into a tree of size n with parallel time $O(\log n + \log k)$.
Fringe analysis studies the distribution of the bottom subtrees and it is still an open problem for parallel algorithms on search trees. To tackle this problem we introduce a new kind of algorithms whose two extreme cases seems to upper and lower bounds the performance of the PVW algorithm.
We extend the fringe analysis to parallel algorithms and we get a rich mathematical structure giving new interpretations even in the sequential case. The process of insertions is modeled by a Markov chain and the coefficients of the transition matrix are related with the expected local behavior of our algorithm. Finally, we show that this matrix has a power expansion over $(n+1)^{-1}$ where the coefficients are the binomial transform of the expected local behavior. This expansion shows that the parallel case can be approximated by iterating the sequential case.

Keywords: *Fringe analysis, Parallel algorithms, 2-3 trees, Binomial transform.*

1 Introduction

Fringe analysis studies the distribution of the bottom subtrees or fringe of trees and has been applied to most search trees in the sequential case [EZG+82,BY95]

We are interested on the fringe analysis of the synchronized parallel algorithms on 2–3 trees designed by W. Paul, U. Vishkin and H. Wagener [PVW83]. This algorithm inserts k keys randomly selected with k processors in time $O(\log n + \log k)$ into a 2–3 tree of size n. The fringe analysis in this case is still open and the main drawback is the reconstructing phase that is composed by waves of synchronized processors which modifies the tree bottom-up.

In this paper we propose a new synchronized parallel algorithm, denoted MacroSplit, that bounds the [PVW83] one in the following sense: the distribution

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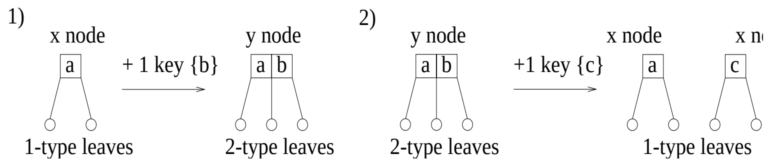

Fig. 1. The transformation of x and y bottom nodes after insertion of one key. In (1) the key b hits a bottom node x (containing the key a). Node x transforms into a node y (having keys a and b). We have $X_{t+1} = X_t - 2$ and $Y_{t+1} = Y_t + 3$. In case (2) the key c hits a bottom node y containing a and b. This the node y splits into 2 nodes x containing a and c respectively, while b is inserted in the parent node recursively. Now $X_{t+1} = X_t + 4$ and $Y_{t+1} = Y_t - 3$.

of the fringe derived from the [PVW83] algorithm is upper and lower bounded by the distribution derived from two extreme cases of our algorithm. The key idea is that our algorithm reconstructs the tree with only one wave meanwhile [PVW83] needs a pipeline of waves.

We have extended the fringe analysis from the sequential case into the parallel case with significant improvements . As later on is showed, the direct extensions of this technique on two concrete cases (the parallel insertion of two and three keys) suggest the inapplicability of this technique on cases greater than these simple ones. We have overcome this drawback with two facts allowing us the analysis of the generic case (the insertion of k keys):

- The random insertion of keys generates a *binomial* distribution on the bottom nodes. This fact allows us the probabilistic analysis of the parallel algorithm.
- The fringe evolution is determined by the expected local behavior of the algorithm. This fact gives a new understanding to fringe analysis.

The rest of the paper is organized as follows. In section 2 we recall the fringe analysis of the sequential case. In section 3 we introduce the MacroSplit algorithms. Section 4 contains the direct extension of the fringe analysis for the parallel introduction of two and three keys. Section 5 contains the analysis of the generic case and section 6 the analysis of two concrete cases of this generic case. Finally section 7 contains the conclusions.

2 Sequential case

The fringe of a tree is composed by the subtrees on the bottom part of a tree. Our fringe is composed by trees of height one. A bottom node with one key is called and x node, and a bottom node with two keys is called an y node. These nodes separate leaves into 1-type leaves if their parents are x nodes and 2-type leaves if their parents are y nodes.

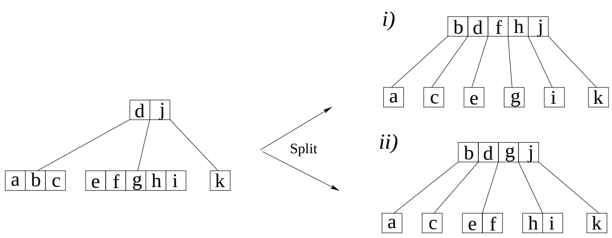

Fig. 2. Choices for MacroSplit rules. In (i) the MaxMacroSplit rule creates a maximum number of splits. In (ii) the MinMacroSplit rule creates the minimum number. Intermediate strategies are allowed.

Let X_t and Y_t be the random variables associated to the number of 1-type leaves and 2-type leaves respectively at the step t. We assume that $X_t + Y_t = n+1$ being n the number of keys of the tree. When a new key falls into a bottom node this node is transformed according the rules given in figure 1. The probability that a key hits a bottom node x is $\frac{X_t}{n+1}$ and for a node y is $\frac{Y_t}{n+1}$. The conditional expectations verify

$$E(X_{t+1} \mid X_t, Y_t, 1) = \frac{X_t}{n+1}(X_t - 2) + \frac{Y_t}{n+1}(X_t + 4) = \left(1 - \frac{2}{n+1}\right) X_t + \frac{4}{n+1} Y_t$$
$$E(Y_{t+1} \mid X_t, Y_t, 1) = \frac{X_t}{n+1}(Y_t + 3) + \frac{Y_t}{n+1}(Y_t - 3) = \frac{3}{n+1} X_t + \left(1 - \frac{3}{n+1}\right) Y_t$$

The expected number of leaves (conditioned to the random insertion of one key) at the step t can be modeled by [EZG$^+$82,BY95]:

$$\begin{pmatrix} E(X_{t+1} \mid 1) \\ E(Y_{t+1} \mid 1) \end{pmatrix} = T_{n,1} \begin{pmatrix} E(X_t \mid 1) \\ E(Y_t \mid 1) \end{pmatrix}$$

As the conditional expectations verify $E(X_{t+1} \mid 1) = E(E(X_{t+1} \mid X_t, Y_t, 1) \mid 1)$ and $E(Y_{t+1} \mid 1) = E(E(Y_{t+1} \mid X_t, Y_t, 1) \mid 1)$ we get from the preceding expression the 1-OneStep transition matrix

$$T_{n,1} = \left(1 + \frac{1}{n+1}\right) I + \frac{1}{n+1} \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \qquad \text{being} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to compare with the parallel case we consider the sequential insertion of k keys given by $T^{\mathsf{Seq}}_{n,k} = T_{n+k-1,1} \cdots T_{n,1}$. It is easy to prove

$$T^{\mathsf{Seq}}_{n,k} = \left(1 + \frac{k}{n+1}\right) I + \frac{k}{n+1} \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} + O\left(\frac{1}{n^2}\right) I$$

k	x node	y node
1	y	xx
2	xx	xy
3	xy	xxx or yy
4	xxx or yy	xxy
5	xxy	$xxxx$ or xyy
6	$xxxx$ or xyy	$xxxy$ or yyy

Table 1. MacroSplit possibilities for x and y bottom nodes once k keys are inserted.

3 MacroSplit parallel insertion algorithms

We introduce a parallel insertion algorithm based on the idea of MacroSplit. On this algorithm an array of ordered keys $a[1\ ..\ ]$ is inserted into a 2-3 tree having n leaves. The MacroSplit insertions algorithm has two main successive phases.

Percolation Phase. In a top-down strategy, the set of keys to be inserted is split into several packets and these packets are routed down. Finally, these packets are attached to the leaves [PVW83].

Reconstruction Phase. In a bottom-up phase the packets attached to the leaves are really inserted and the tree is reconstructed. This reconstruction is based in just *one unique* wave moving bottom up. First, the packets are incorporated at the bottom internal nodes of the tree. In successive steps the wave moves up, decreasing the depth one unit at each time. The evolution of this unique wave needs the usage of rules so called MacroSplit rules (see Figure 2).

The MacroSplit algorithm can be seen as a "height level" description of the parallel insertion algorithm given by W. Paul, U. Vishkin and H. Wagener in [PVW83] which take place by splitting a MacroSplit step into several more basic steps chained together in a pipeline.

Let us see why we have several MacroSplit algorithms for a large k. At most, k keys can reach a node. If the node stores more than two keys, it must split using a MacroSplit rule. Table 1 show us several split possibilities for x and y bottom nodes. For instance, the first row show us the splits of the x and y nodes when $k = 1$(see Figure 1). In this case there is just one possibility. The fourth row show us how x and y nodes can be split when $k = 4$. In this case a bottom node x can be split into 3 nodes x or into 2 nodes y. Later on we will consider two extreme cases. The MaxMacroSplit algorithm will maximize the number of splits at each step and the MinMacroSplit algorithm will minimize this number. When $k = 1$ or 2 both algorithms coincides (see table 1).

Consider that at the $t+1$ step k random keys (we asume a uniform distribution of them) fall in parallel into a fringe with X_t leaves of 1-type and Y_t leaves 2-type such that $X_t + Y_t = n + 1$. The expected values of X_{t+1} and Y_{t+1} after the insertions depends on two facts.

$(\cdot,\cdot)$	$P(\cdot,\cdot)$	$E(X_{t+1} \mid X_t, Y_t, 2, (.,.))$	$E(Y_{t+1} \mid X_t, Y_t, 2, (.,.))$
(x,x)	$\frac{X_t}{n+1}\frac{2}{n+1}$	$X_t + 2$	Y_t
(x_1,x_2)	$\frac{X_t}{n+1}\frac{X_t-2}{n+1}$	$X_t - 4$	$Y_t + 6$
(x,y)	$2\frac{X_t}{n+1}\frac{Y_t}{n+1}$	$X_t + 2$	Y_t
(y,y)	$\frac{Y_t}{n+1}\frac{3}{n+1}$	$X_t + 2$	Y_t
(y_1,y_2)	$\frac{Y_t}{n+1}\frac{Y_t-3}{n+1}$	$X_t + 8$	$Y_t - 6$

Table 2. Parallel insertion of two keys

- The concrete form of the MacroSplit algorithm. This algorithm explicites how many leaves of 1-type and 2-type will be generated by bottom nodes when they receive some number of keys.
- The preceding values of X_t and Y_t.

We deal with a Markov chain and the evolution can be analyzed through the so called k-OneStep transition matrix $T_{n,k}$

$$\begin{pmatrix} E(X_{t+1} \mid k) \\ E(Y_{t+1} \mid k) \end{pmatrix} = T_{n,k} \begin{pmatrix} E(X_t \mid k) \\ E(Y_t \mid k) \end{pmatrix}$$

4 Parallel insertion of 2 and 3 keys

In this section we compute $T_{n,2}$ and $T_{n,3}$ following directly the technique applied before to sequential insertions [EZG+82] and we discuss the viability of this approach.

4.1 Direct extensions

First, let us consider the case $k = 2$. We have only one MacroSplit algorithm (see Table 1). The expected number of leaves is characterized by 2-OneStep $T_{n,2}$ transition matrix:

$$\begin{pmatrix} E(X_{t+1} \mid 2) \\ E(Y_{t+1} \mid 2) \end{pmatrix} = T_{n,2} \begin{pmatrix} E(X_t \mid 2) \\ E(Y_t \mid 2) \end{pmatrix}.$$

We compute the probabilities of the different splits by an exhaustive case analysis (see Table 2). As at most two keys can reach the same bottom node, we have no election in the split, *i.e.* the transformation of bottom nodes is unique (second row of table 1). Both keys can be either at the same bottom node or at different bottom nodes, and in each case bottom nodes can be of type x or y. Let $P(x, x)$ be the probability that both keys reach the same x node, $P(x_1, x_2)$ the probability to reach different x nodes and so on for the remainder probabilities $P(x, y)$ and $P(y_1, y_2)$. We denote the generic case as $P(\cdot, \cdot)$, being $(.,.)$ the generic pair of nodes accessed.

As $E(X_{t+1} \mid 2) = E(E(X_{t+1} \mid X_t, Y_t, 2))$ we compute the expected number of 1-type leaves as $E(X_{t+1}|X_t, Y_t, 2) = \sum_{(\cdot,\cdot)} P(\cdot, \cdot)\, E(X_{t+1}|X_t, Y_t, 2, (.,.))$ being $E(X_{t+1}|X_t, Y_t, 2, (.,.))$ the expected number of 1-type leaves when 2 keys reach node $(\cdot, \cdot)$ conditioned to X_t and Y_t. For instance, if both keys reach different x nodes then it holds

$$P(x_1, x_2) = \frac{X_t}{n+1} \frac{X_t - 2}{n+1}$$

and $E(X_{t+1}|X_t, Y_t, 2, (x1, x2)) = X_t - 4$ (table 2 contains the other values). In the appendix we give the proof of the following lemma.

Lemma 1. *The conditional expectations verify*

$$E(X_{t+1} \mid X_t, Y_t, 2) = \left(1 - \frac{4}{n+1} + \frac{12}{(n+1)^2}\right) X_t + \left(\frac{8}{n+1} - \frac{18}{(n+1)^2}\right) Y_t$$
$$E(Y_{t+1} \mid X_t, Y_t, 2) = \left(1 - \frac{6}{n+1} + \frac{18}{(n+1)^2}\right) Y_t + \left(\frac{6}{n+1} - \frac{12}{(n+1)^2}\right) X_t$$

As the conditional expectations are linear in X_t and Y_t and $E(X_{t+1} \mid 2) = E(E(X_{t+1} \mid X_t, Y_t, 2))$, $E(Y_{t+1} \mid 2) = E(E(Y_{t+1} \mid X_t, Y_t, 2))$ we have:

Lemma 2. *The 2-OneStep transition matrix is:*

$$T_{n,2} = \left(1 + \frac{2}{n+1}\right) I + \frac{2}{n+1} \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} + \frac{1}{(n+1)^2} \begin{pmatrix} 12 & -18 \\ -12 & 18 \end{pmatrix}$$

Consider briefly the case $k = 3$. Now there are two possibilities (third row of table 1). We have selected the second transformation. This corresponds to the MinMacroSplit algorithm. As before, an exhaustive case analysis give us:

Lemma 3. *In the case of the MinMacroSplit algorithm, the 3-OneStep transition matrix $T_{n,3}$ is:*

$$\left(1 + \frac{3}{n+1}\right) I + \frac{3}{n+1} \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} + \frac{3}{(n+1)^2} \begin{pmatrix} 12 & -18 \\ -12 & 18 \end{pmatrix} + \frac{1}{(n+1)^3} \begin{pmatrix} -48 & 54 \\ 48 & -54 \end{pmatrix}$$

4.2 Discussion of the cases 2 and 3

Based on the preceding cases can point several facts and questions:

1. The exhaustive case analysis (generalizing the sequential approach [EZG+82]) for larger k, $k = 5, 6, \ldots$, becomes intractable.

2. For $k = 1, 2, 3$ the expectations $E(X_{t+1} \mid X_t, Y_t, k)$ and $E(X_{t+1} \mid X_t, Y_t, k)$ are linear in X_t and Y_t. It is unclear why non-linear terms always disappears. Note that we assume this point of view in the equation k-OneStep transition matrix $T_{n,k}$.
3. The intuitive meaning of the coefficients appearing in the expectations is unclear. For instance, the term $1 - \frac{4}{n+1} + \frac{12}{(n+1)^2}$ appearing in $E(X_{t+1} \mid X_t, Y_t, 2)$ in lemma 1 does not have any direct explanation in terms of the MacroSplit algorithm.
4. By *local behavior* of the algorithm we mean what happens when i keys hit just *one* bottom node x or y (table 1) . By *global behavior* we mean the evolution of X_t and Y_t. The previous exhaustive analysis does not give a clear cut between the *local* and the *global behavior* of the MacroSplit algorithm.
5. Note that
   - Lemmas 2 and 3 can be envisaged as a *power expansion* over $(n+1)^{-1}$ of the transition matrix.
   - The matrices appearing when $k = 2$ also appears for $k = 3$ (see lemmas 2 and 3).

   This suggest us a power expansion of the k-OneStep of the form

$$T_{n,k} = \left(1 + \frac{k}{n+1}\right) I + \frac{\gamma_1(k)}{n+1} \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} + \frac{\gamma_2(k)}{(n+1)^2} \begin{pmatrix} 12 & -18 \\ -12 & 18 \end{pmatrix} + \frac{\gamma_3(k)}{(n+1)^3} \begin{pmatrix} -48 & 54 \\ 48 & -54 \end{pmatrix} + \cdots$$

   Moreover, a little bit of though suggest us $\gamma_i(k) = \binom{k}{i} \cdots$
6. The different coefficients appearing into the matrices reflect the behavior of the MacroSplit algorithm. We search for a precise meaning of this intuitive fact.

In the following we solve all these questions.

5 Behavior of the MacroSplit algorithms

In order to study the expected behavior of an x or y node belonging to a fringe of $n + 1$ leaves when k keys are inserted at a given step, we need to know the characteristics of the MacroSplit algorithm we are using.

5.1 Local behavior

We would like to know how many 1-type and 2-type leaves are generated when i keys fall in the same step into a unique node x or y. To deal with this fact we introduce the following definition.

Definition 4. At the bottom level, the local behavior of the MacroSplit algorithm is given by the following functions:

- The $\mathcal{X}_x(i)$ is the number of 1-type leaves after the insertion of i keys into a unique x node (for instance, $\mathcal{X}_x(0) = 2$, $\mathcal{X}_x(1) = 0, \ldots$). In the same way, $\mathcal{X}_y(i)$ is the number of 1-type leaves after the insertion of i keys into an y node(for instance, $\mathcal{X}_y(0) = 0$, $\mathcal{X}_y(1) = 4, \ldots$).
- Dually, $\mathcal{Y}_x(i)$ is the number of 2-type leaves after the insertion of i keys into an x node. Finally, $\mathcal{Y}_y(i)$ is the number of 2-type leaves after the insertion of i keys into an y node.

These coefficients verify $\mathcal{X}_x(i) + \mathcal{Y}_x(i) = 2 + i$ and $\mathcal{X}_y(i) + \mathcal{Y}_y(i) = 3 + i$.

5.2 Distribution function

Assume that random k keys fall (in parallel) into a fringe having $n + 1$ leaves. First of all, let us isolate just one bottom node x and one key to insert. Fixed x, it has two leaves, and one new key can be inserted into this node in two different positions (corresponding to the left of each leaf). Therefore just one key hits a node x with probability $\frac{2}{n+1}$.

Now we consider what happens with this node x when k keys are inserted. Let N_x the a random variable denoting the number of keys falling into a fixed bottom node x. As the set of k keys is random, this variable follows the binomial distribution,

$$P\{N_x = i\} = \binom{k}{i}\left(\frac{2}{n+1}\right)^i\left(1 - \frac{2}{n+1}\right)^{k-i} = b\Big(i, k, \frac{2}{n+1}\Big)$$

such that $b\,(i, k, p) = \binom{k}{i}p^i\,(1-p)^{k-i}$. Recall that the expected value is kp.

5.3 Expected local behavior

The number of 1-type leaves generated by the keys falling into a unique node x is given by the random variable $X_x = \mathcal{X}_x(N_x)$. The expected number of 1-type leaves generated by one x bottom node when a batch of k keys is inserted into a fringe having $n + 1$ leaves is:

$$E(X_x \mid k) = \sum_{i=0}^{k} P\{N_x = i\}\mathcal{X}_x(i) = \sum_{i=0}^{k} b\Big(i, k, \frac{2}{n+1}\Big)\mathcal{X}_x(i)$$

The number of 2-type leaves generated by the node x is $Y_x = \mathcal{Y}_x(N_x)$, then the expected value is is

$$E(Y_x \mid k) = \sum_{i=0}^{k} b\Big(i, k, \frac{2}{n+1}\Big)\mathcal{Y}_x(i)$$

Let us fix a bottom node y. In this case, one key hits this node with probability $\frac{3}{n+1}$. Let N_y be another random variable denoting the number of keys falling into this node y, clearly $P\{N_y = i\} = b\Big(i, k, \frac{3}{n+1}\Big)$. In this case we have the random variables $X_y = \mathcal{X}_y(N_y)$ and $Y_y = \mathcal{Y}_y(N_y)$ having the expected values

$$E(X_y \mid k) = \sum_{i=0}^{k} b\Big(i, k, \frac{3}{n+1}\Big)\mathcal{X}_y(i) \qquad E(Y_y \mid k) = \sum_{i=0}^{k} b\Big(i, k, \frac{3}{n+1}\Big)\mathcal{Y}_y(i)$$

Note that these expected values depend of the concrete local behavior of the algorithm. The expected number of leaves generated by just one bottom node when k random keys are inserted in parallel into a fringe having $n+1$ is:

$$E(X_x + Y_x \mid k) = 2(1 + \frac{k}{n+1}) \qquad \text{and} \qquad E(X_y + Y_y \mid k) = 3(1 + \frac{k}{n+1})$$

5.4 Global behavior

We relate the local behavior with the global one by means of the matrix transition:

Definition 5. Given a fringe with $n+1$ leaves and a MacroSplit algorithm, we define the k-OneStep transition matrix as:

$$T_{n,k} = \begin{pmatrix} \frac{1}{2}E(X_x \mid k) & \frac{1}{3}E(X_y \mid k) \\ \frac{1}{2}E(Y_x \mid k) & \frac{1}{3}E(Y_y \mid k) \end{pmatrix}$$

The proof of the following lemma is given in the appendix.

Lemma 6. *Given a fringe with X_t leaves of 1-type and Y_t leaves of 2-type, when k random keys are inserted into it in one step we have*

$$\begin{pmatrix} E(X_{t+1} \mid X_t, Y_t, k) \\ E(Y_{t+1} \mid X_t, Y_t, k) \end{pmatrix} = T_{n,k} \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$

The proof of the following theorem is given in the appendix.

Theorem 7. *When k random keys are inserted in one step we have:*

$$\begin{pmatrix} E(X_{t+1} \mid k) \\ E(Y_{t+1} \mid k) \end{pmatrix} = T_{n,k} \begin{pmatrix} E(X_t \mid k) \\ E(Y_t \mid k) \end{pmatrix}$$

From the note 5 of the section 4.2 in which we have conjectured a power expansion form for the transition matrix, it will be interesting to have a k-OneStep transition matrix (definition 5) like $T_{n,k} = \left(1 + \frac{k}{n+1}\right) I + \cdots$. We can prove:

Lemma 8. *Let I be the two dimensional identity matrix, the k-OneStep verifies:*

$$T_{n,k} = \left(1 + \frac{k}{n+1}\right) I + \begin{pmatrix} -\frac{1}{2}E(Y_x \mid k) & \frac{1}{3}E(X_y \mid k) \\ \frac{1}{2}E(Y_x \mid k) & -\frac{1}{3}E(X_y \mid k) \end{pmatrix}$$

5.5 Power expansion on the transition matrix

Let us recall the *binomial transform* $\mathcal{B}$ recently developed by P. Poblete, J. Munro and Th. Papadakis [PMP95]. Let $\langle F_i \rangle_{i \geq 0}$ be a sequence of real numbers, the binomial transform is the sequence $\langle \hat{F}_j \rangle_{j \geq 0}$ defined as

$$\hat{F}_j = \mathcal{B}_j F_i = \sum_{i=0}^{j} (-1)^i \binom{j}{i} F_i.$$

This transformation verifies $F_i = \mathcal{B}_i \hat{F}_j$. In the following we will use the following weighted form of the binomial transforms of $\langle \mathcal{Y}_x(i) \rangle_{i \geq 0}$ and $\langle \mathcal{X}_y(i) \rangle_{i \geq 0}$:

Definition 9. Let consider the coefficients $\alpha_j = -2^{j-1}\hat{\mathcal{Y}}_x(j)$ and $\beta_j = -3^{j-1}\hat{\mathcal{X}}_y(j)$.

Let us develop the relationship of the preceding coefficients with the local expected values of the k-OneStep appearing in the lemma 8. The proof of the following lemma is given in the appendix.

Lemma 10.

$$E(Y_x \mid k) = \mathcal{B}_k\left(\left(\frac{2}{n+1}\right)^j \mathcal{B}_j\mathcal{Y}_x(i)\right) = -2\sum_{j=0}^{k}\frac{(-1)^j}{(n+1)^j}\binom{k}{j}\alpha_j$$

$$E(X_y \mid k) = \mathcal{B}_k\left(\left(\frac{3}{n+1}\right)^j \mathcal{B}_j\mathcal{X}_y(i)\right) = -3\sum_{j=0}^{k}\frac{(-1)^j}{(n+1)^j}\binom{k}{j}\beta_j$$

From lemmas 6 and 10 we get the following expansion

Theorem 11. *The k-OneStep transition matrix can be rewritten as*

$$T_{n,k} = \left(1+\frac{k}{n+1}\right) I + \sum_{j=0}^{k}\frac{(-1)^j}{(n+1)^j}\binom{k}{j}\begin{pmatrix}\alpha_j & -\beta_j\\ -\alpha_j & \beta_j\end{pmatrix},$$

Note that $T_{n,k} = T^{\mathsf{Seq}}_{n,k} + O\left(1/n^2\right) I$.

6 Two extreme MacroSplit algorithms

We have shown that the k-OneStep transition matrix depends of the concrete MacroSplit algorithm. In this section we develop two extreme cases of this algorithm: one denoted MaxMacroSplit algorithms that makes the maximum number of splits and creates the maximum number of x nodes and another denoted MinMacroSplit algorithm that makes the minimum number of splits and creates the maximum number of y nodes. These two extreme cases seems to bound the behavior of the whole pipeline in the W. Paul, U. Vishkin and H. Wagener [PVW83] insertion algorithm.

6.1 The MaxMacroSplit and MinMacroSplit algorithms

Assume that an even i number of keys are attached to a node x ($i = 6$ in the case 1 of the figure 3). This wide node splits by yielding $i+2$ 1-type leaves (8 in the preceding case) and 0 2-type leaves. Then $\mathcal{X}_x(i) = i+2$ and $\mathcal{Y}_x(i) = 0$. On the other hand, an odd number i of keys are attached ($i = 7$ in case 2 of the figure 3). In this case the split only creates one node y, then $\mathcal{Y}_x(i) = 3$ and $\mathcal{X}_x(i) = i-1$ (3 and 6 respectively in the figure). Note that $\mathcal{X}_x(i)+\mathcal{Y}_x(i) = i+2$. We summarize the previous paragraph into the following lemma.

Lemma 12. *The MaxMacroSplit algorithm has the following characterization: (1) The* local behavior *is given by:*

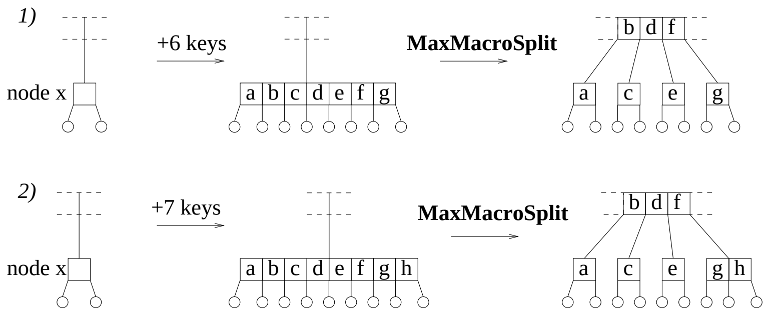

Fig. 3. Application of MaxMacroSplit rule on a node x

– *For even i we have $\mathcal{X}_x(i) = i + 2$, $\mathcal{Y}_x(i) = 0$, $\mathcal{X}_y(i) = i$, $\mathcal{Y}_y(i) = 3$.*
– *For odd i we have $\mathcal{X}_x(i) = i - 1$, $\mathcal{Y}_x(i) = 3$, $\mathcal{X}_y(i) = i + 3$, $\mathcal{Y}_y(i) = 0$.*

(2) The expected local behavior *is*

$$E(X_x \mid k) = kp + \frac{1}{2} + \frac{3}{2}(q-p)^k \quad E(Y_x \mid k) = \frac{3}{2} - \frac{3}{2}(q-p)^k \quad \textit{for} \quad p = \frac{2}{n+1}$$
$$E(X_y \mid k) = kp + \frac{3}{2} - \frac{3}{2}(q-p)^k \quad E(Y_y \mid k) = \frac{3}{2} + \frac{3}{2}(q-p)^k \quad \textit{for} \quad p = \frac{3}{n+1}$$

(3) The power expansion *verifies $\alpha_0 = \beta_0 = 0$, $\beta_1 = 4$. For $j > 0$ we have $\alpha_j = -3\,4^{j-1}$ and for $j > 1$ we have $\beta_j = -3\,6^{j-1}$.*

When we consider a minimum number of splits we have the following

Lemma 13. *The MinMacroSplit algorithm has the following characterization:*
(1) The local behavior *is given by:*

– *For $i \mod 3 = 0$ we have $\mathcal{X}_x(i) = 2$, $\mathcal{Y}_x(i) = i$, $\mathcal{X}_y(i) = 0$, $\mathcal{Y}_y(i) = i + 3$.*
– *For $i \mod 3 = 1$ we have $\mathcal{X}_x(i) = 0$, $\mathcal{Y}_x(i) = i+2$, $\mathcal{X}_y(i) = 4$, $\mathcal{Y}_y(i) = i-1$.*
– *For $i \mod 3 = 2$ we have $\mathcal{X}_x(i) = 4$, $\mathcal{Y}_x(i) = i-2$, $\mathcal{X}_y(i) = 2$, $\mathcal{Y}_y(i) = i+1$.*

(2) Let be
$$\phi = \mathcal{R}e\left(\frac{2 - 3p + p\sqrt{3}\mathbf{i}}{2}\right)^k \qquad \textit{and} \qquad \varphi = \sqrt{3}\,\mathcal{I}m\left(\frac{2 - 3p + p\sqrt{3}\mathbf{i}}{2}\right)^k$$

The expected local behavior *is determined by:*

$$E(X_x \mid k) = 2 - \frac{4}{3}\varphi \qquad E(Y_x \mid k) = pk + \frac{4}{3}\varphi \qquad \textit{for} \quad p = \frac{2}{n+1}$$
$$E(X_y \mid k) = 2 - 2\phi + \frac{2}{3}\varphi \quad E(Y_y \mid k) = pk + 1 + 2\phi - \frac{2}{3}\varphi \quad \textit{for} \quad p = \frac{3}{n+1}.$$

(3) For $j > 2$, the power expansion coefficients of the MinMacroSplit algorithm verify $\alpha_{j+6} = 12^3\alpha_j$ and $\beta_{j+6} = 12^3\beta_j$

6.2 Relationship with PVW's algorithm

Let us see how the MaxSplit and MinSplit algorithms bound the fringe behavior of the insertion algorithm given in [PVW83]. On it, a *macro step* contains the whole insertion of the k keys. Let X_t^{PVW}, Y_t^{PVW} the fringe distribution before the pipeline starts and let be X_{t+1}^{PVW}, Y_{t+1}^{PVW} be the fringe once the pipeline has finished. A rough bound is given in the following conjecture.

Conjecture 14. *Let $X_t^{MaxSplit}$, $Y_t^{MaxSplit}$ be the fringe in the MaxMacroSplit algorithm. Let $X_t^{MinSplit}$, $Y_t^{MinSplit}$ be the fringe in the MinMacroSplit algorithm and Let X_t^{PVW}, Y_t^{PVW} be the fringe in the macro step algorithm of W. Paul, U. Vishkin and H. Wagener, we have:*

$$E(X_t^{MinSplit} \mid k) \leq E(X_t^{PVW} \mid k) \leq E(X_t^{MaxSplit} \mid k)$$
$$E(Y_t^{MaxSplit} \mid k) \leq E(Y_t^{PVW} \mid k) \leq E(Y_t^{MinSplit} \mid k)$$

7 Conclusion

We have analyzed the MacroSplit parallel insertion algorithms (Theorem 7) and we have proved that the coefficients of the k-OneStep, determining the global behavior of the algorithm, are given by the expected local behavior. We have developed the power expansion (theorem 11) proving that the MacroSplit algorithm can be approximated by the iterative sequential algorithm with an error determined by $O(1/n^2)$ (being n the size of the tree). The coefficients of the expansion are proportional to the binomial transform of the expected local behavior

We have conjectured (conjecture 14) that the [PVW83] algorithm is bounded by the two extreme algorithms MaxMacroSplit and MinMacroSplit and we have computed (lemmas 12 and 13) the main values of these algorithms. In the limiting case (very large trees) all these algorithms have the same performance.

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A Appendix

A.1 Proof of lemma 1

The conditional expectation $E(X_{t+1}|X_t, Y_t, 2)$ is:

$$
\begin{aligned}
&\sum_{(\cdot,\cdot)} P(\cdot,\cdot)\ E(X_{t+1}|X_t, Y_t, 2, (.\ ,.)) \\
=&\frac{1}{(n+1)^2}\Big(2X_t(X_t+2) + X_t(X_t-2)(X_t-4) + 2X_tY_t(X_t+2) \\
&\qquad + 3Y_t(X_t+2) + Y_t(Y_t-3)(X_t+8)\Big) \\
=&X_t + \frac{1}{(n+1)^2}\Big(12X_t - 4X_t^2 + 4X_tY_t + 8Y_t^2 - 18Y_t\Big) \\
=&X_t + \frac{1}{(n+1)^2}\Big(12X_t - 4X_t(X_t+Y_t) + 8Y_t(X_t+Y_t) - 18Y_t\Big)
\end{aligned}
$$

The Y_{t+1} term has a similar development.

A.2 Proof of lemma 6

Let us consider a fringe having X_t leaves of 1-type and Y_t leaves of 2-type and $X_t + Y_t = n + 1$. In this fringe, the number of x bottom nodes is $X_t/2$. The number of y bottom nodes is $Y_t/3$. Now we insert k random keys in just one step and we are interested in the value of X_{t+1}. Let $\mathcal{N}_{x,i}$ be the number of x nodes getting i keys and let $\mathcal{N}_{y,i}$ be the number of y nodes getting i keys. We have

$$X_{t+1} = \sum_{i=0}^{k} \mathcal{N}_{x,i} \cdot \mathcal{X}_x(i) + \sum_{i=0}^{k} \mathcal{N}_{y,i} \cdot \mathcal{X}_y(i)$$

Recall that X_t and Y_t are fixed. As an x node gets i keys with probability $P\{N_x = i\}$ and the number of x nodes is $\frac{1}{2}X_t$, the random variable $\mathcal{N}_{x,i}$ follows the binomial distribution

$$P\{\mathcal{N}_{x,i} = j \mid X_t, Y_t, k\} = b\Big(j, \frac{1}{2}X_t, P\{N_x = i\}\Big)$$

Then the expected number of x nodes receiving exactly i keys each one is $E(\mathcal{N}_{x,i} \mid X_t, Y_t, k) = P\{N_x = i \mid k\}\frac{X_t}{2}$. Similarly, the expected number of y nodes is $E(\mathcal{N}_{y,i} \mid X_t, Y_t, k) = P\{N_y = i \mid k\}\frac{Y_t}{3}$.

We study the expected behavior of X_{t+1} when k keys are inserted at random.

$$
\begin{aligned}
E(X_{t+1} \mid X_t, Y_t, k) &= \sum_{i=0}^{k} E(\mathcal{N}_{x,i} \mid X_t, Y_t, k)\mathcal{X}_x(i) + \sum_{i=0}^{k} E(\mathcal{N}_{y,i} \mid X_t, Y_t, k)\mathcal{X}_y(i) \\
&= \frac{X_t}{2}\sum_{i=0}^{k} P\{N_x = i \mid k\}\mathcal{X}_x(i) + \frac{Y_t}{3}\sum_{i=0}^{k} P\{N_y = i \mid k\}\mathcal{X}_y(i) \\
&= E(X_x \mid k)\frac{X_t}{2} + E(X_y \mid k)\frac{Y_t}{3}
\end{aligned}
$$

The computation for $E(Y_{t+1} \mid X_t, Y_t, k)$ is similar.

A.3 Proof of theorem 7

From the preceding lemma we have

$$E(X_{t+1} \mid X_t, Y_t, k) = E(X_x \mid k)\frac{X_t}{2} + E(X_y \mid k)\frac{Y_t}{3}$$

As $E(X_t + 1 \mid k) = E(E(X_{t+1} \mid X_t, Y_t, k) \mid k)$ we have

$$E(X_t + 1 \mid k) = \frac{1}{2}E(E(X_x \mid k)X_t \mid k) + \frac{1}{3}E(E(X_y \mid k)Y_t \mid k)$$

As X_x and X_t are independent $E(E(X_x \mid k)X_t \mid k) = E(X_x \mid k)E(X_t \mid k)$ and the proof is done.

A.4 Proof of lemma 10

Recall that

$$E(Y_x \mid \ell) = \sum_{\ell=0}^{k} P\{X_x = i\}\mathcal{Y}_x(i) = \sum_{\ell=0}^{k} \binom{k}{\ell}\left(\frac{2}{n+1}\right)^{\ell}\left(1 - \frac{2}{n+1}\right)^{k-\ell}\mathcal{Y}_x(i)$$

Consider the sequence $\langle E(Y_x \mid \ell)\rangle_{\ell \geq 0}$, given $p + q = 1$ the binomial transform verifies $\mathcal{B}_j \sum_i \binom{\ell}{i} p^i q^{\ell-i} F_i = p^j \hat{F}_j$ therefore:

$$\hat{E}(Y_x \mid j) = \mathcal{B}_j E(Y_x \mid \ell) = \left(\frac{2}{n+1}\right)^j \mathcal{B}_j \mathcal{Y}_x(\ell) = \left(\frac{2}{n+1}\right)^j \hat{\mathcal{Y}}_x(j)$$

Now we apply the property $F_k = \mathcal{B}_k \mathcal{B}_j F_\ell$ and

$$E(Y_x \mid k) = \mathcal{B}_k \hat{E}(Y_x \mid j) = \mathcal{B}_k \mathcal{B}_j E(Y_x \mid \ell) = \mathcal{B}_k \left(\left(\frac{2}{n+1}\right)^j \mathcal{B}_j \mathcal{Y}_x(i)\right)$$

Using linearity $\mathcal{B}_k \gamma F_j = \gamma \mathcal{B}_k F_j$ and $\alpha_j = -2^{j-1}\mathcal{B}_j \mathcal{Y}_x(i)$ we have

$$\begin{aligned} E(Y_x \mid k) &= 2\mathcal{B}_k\left(\left(\frac{1}{n+1}\right)^j 2^{j-1}\mathcal{B}_j\mathcal{Y}_x(i)\right) = 2\mathcal{B}_k\left(\left(\frac{1}{n+1}\right)^j \alpha_j\right) \\ &= -2\sum_{j=0}^{k}(-1)^j \binom{k}{j}\frac{\alpha_j}{(n+1)^j} \end{aligned}$$

The case $E(x_y \mid k)$ is quite similar.